

Pol zweiter Ordnung bei $p=0$ ergibt. Bei Ausrechnung von (62) muß dann auch der Beitrag von $\exp\{p w\} = 1 + p w \dots$ zum Residuum $\bar{R}_0(w)$ des ganzen Integranden berücksichtigt werden. Es wird

$$\bar{R}_0(w) = \frac{-\sum \beta_i / u_i}{\sum \alpha_j / u_j} + \frac{\sum \beta_i \cdot \sum \alpha_j / u_j^2}{(\sum \alpha_j / u_j)^2} + w \frac{\sum \beta_i}{\sum \alpha_j / u_j}. \quad (71)$$

Hierbei ist vorausgesetzt, daß die im Nenner auftretende Summe ungleich Null ist. Demgemäß hat man in (63) und mit $w=0$ in (64) zu ersetzen

$$R_0 + R_q \exp[-\bar{u}_q w] \rightarrow \bar{R}_0(w). \quad (72)$$

Diese Relation kann man wie gesagt auch als Grenzübergang erhalten, indem man $\sum \alpha_j = 1 + \delta a$ setzt und δa oder das zugehörige \bar{u}_q gegen Null gehen läßt, vgl. den Übergang zu (32 a). Bei der Rücktransformation von (66) ist entsprechend auch der dortige Bruch $= 1 - (p/u_j) \dots$ zu berücksichtigen.

Dies ergibt in der eckigen Klammer von (69) zusätzlich als Summanden

$$- \frac{1}{u_j} \frac{\sum \beta_i}{\sum (\alpha_i / u_i)},$$

während in der Summe über l der Term mit $l=q$ wegfällt. Das erhält man auch sofort, wenn man R_q nach (61) einsetzt, für $\bar{u}_q \rightarrow 0$.

Wenn alle α_j negativ sind, folgt aus dem Verlauf von (60), daß die obigen Formeln (63) bis (70) gültig bleiben und daß jetzt $\bar{u}_j > u_j$ ist. Wenn die α_j teils positiv, teils negativ sind, können die als Nullstellen definierten \bar{u}_j und die zugehörigen R_j komplex sein. Jedoch werden die Summen in (63) und (70) wegen des paarweisen Auftretens reell, wie es sein muß.

Herrn cand. MÜHLIG danke ich für fördernde Diskussionen und Hinweise, Herrn Prof. A. KLEMM für anregende und fördernde Korrespondenz.

Composite States for Fourfermion Interactions by the Method of Broken Symmetries*

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Composite states of quasiparticles are studied in the frame of Bopp's quantum theory of elementary particles in a lattice space. By solving the homogeneous Bethe-Salpeter equations in the combined chain and ladder approximation the theory is found to contain a massless pseudoscalar particle (the Goldstone boson), a scalar particle of mass $2m$ and an axialvector particle of mass $2m-d$, $d > 0$, where m is the mass of the quasiparticles and d a small quantity depending on the number of lattice points (or equivalently the cutoff). The method of summation of chain and ladder diagrams by means of the Fierz formula is treated in some detail. Analogies with the model of Nambu and Jona-Lasinio are pointed out. Finally some remarks on the scattering problem are added.

1. Introduction

In several papers BOPP¹⁻³ has recently formulated a quantum theory of elementary particles in the frame of a three-dimensional lattice space. This theory is completely free of mathematical existence problems (which in any way appear to be physically irrelevant) and offers a direct physical explanation for the cutoff necessary in making a non-renormalizable theory finite. Ordinary three-dimensional space is considered to be a cubic lattice of Z^3

lattice points, Z being a huge but finite number ($\sim 10^{83}$). Lorentz invariance, however, is regained only in the limit $Z \rightarrow \infty$. In particular Bopp applied this approach to a one-field theory of the Heisenberg type. A fundamental fermion or "urfermion" field is assumed to exist whose motion is understood to be such that the creation or annihilation of an urfermion at any lattice point corresponds to the annihilation or creation of an urfermion at a neighbouring lattice point. The fourfermion interaction operator chosen is the same as that of Heisenberg's

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¹ F. BOPP, Z. Physik **200**, 117, 133, 142 [1967].

² F. BOPP, S.-B. Bayer. Akad. Wiss., Math.-Naturw. Klasse **1967**, 13.

³ E. G. WEIDEMANN and F. BOPP, Z. Physik **204**, 311 [1967].



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equation⁴ which is a particularly good candidate in view of its high degree of symmetry. As a result of their self-interaction the urfermions acquire a mass. This mass or rather the ground-state energy of the urfermions was calculated by WEIDEMANN and BOPP³ in the wellknown Hartree-Fock approximation by using a Ritz variational procedure. Later WEIDEMANN⁵ has shown that fieldtheoretical techniques such as the use of Feynman rules can easily be transcribed into Bopp's lattice space fomulation.

The object of the present investigation was the calculation of composite states. If the massive urfermion conglomerate or dressed urfermion or quasiparticle in the Dirac-sea of urfermions is identified with the nucleon, then one would expect composite states of quasiparticle-antiquasiparticle pairs to be identifiable with other physical states such as a pseudoscalar meson. These composite states emerge in a natural way as poles of the complete one-particle Green's function, so that their calculation is in fact intimately connected with that of the quasiparticle selfenergy. The nonzero quasiparticle mass results from choosing a symmetry-breaking solution of the eigenvalue problem. In fact one could start from the assumption that the equilibrium of the urfermion system is disturbed in some way (e. g. by a current-source which is set equal to zero at the end of the calculations) such that the disturbance

manifests itself in the degeneracy of the vacuum with respect to a symmetry operation. This assumption suffices in principle to yield e. g. the mass of quasiparticles and their composite states. Then one of the latter has to be a massless bound state as dictated by the Goldstone theorem.

Composite states may be studied either in the frame of the Bethe-Salpeter equation or with the help of Schwinger's functional technique. We shall choose the first method here. In Sect. 2 we recapitulate some basic concepts of Bopp's lattice space theory and determine the Hartree-Fock equation of the quasiparticle selfenergy which is essential for all subsequent calculations. In Sect. 3 we seek bound states as solutions to the homogeneous Bethe-Salpeter equation. We also discuss in some detail the method of summation of mixed chain and ladder diagrams with the help of the Fierz formula. Finally we make some remarks with regard to the scattering problem.

2. Hartree-Fock Equation of Quasiparticle Selfenergy

In Bopp's lattice space theory the Hamiltonian has the form (setting $\hbar = c = 1$)

$$H = H_B + H_W,$$

where

$$\begin{aligned} H_B &= -i \frac{\sqrt{Z}}{2l} \sum_{\mathbf{n}} \psi^+(\mathbf{n}) \varrho_1 \sum_{i=1}^3 \sigma_i \{ \psi(\mathbf{n} + \mathbf{e}_i) - \psi(\mathbf{n} - \mathbf{e}_i) \} \\ H_W &= W \sum_{\mathbf{n}} (\psi^+(\mathbf{n}) \bar{O}^j \psi(\mathbf{n})) (\psi^+(\mathbf{n}) \bar{O}_j \psi(\mathbf{n})), \end{aligned} \quad (2.1)$$

and

$$O^j \cdot \bar{O}_j = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \varrho_1 \cdot \varrho_1.$$

Here $\mathbf{n} = (n_1, n_2, n_3)$, n_i integral, denotes lattice vectors (modulo Z), \mathbf{e}_i a unit vector and W a coupling constant. ψ, ψ^+ are respectively annihilation and creation operators of urfermions, and l is a fundamental length of nuclear magnitude. l may be expressed as $a\sqrt{Z}$ or A/\sqrt{Z} where a is the lattice constant (i. e. the distance between adjacent lattice points) and A the length of an edge of the world cube. Bopp's matrices σ, ϱ may be reexpressed in terms of the more familiar Dirac matrices. One finds

$$\begin{aligned} \gamma^0 &= \varrho_3, \quad \gamma^i = i \varrho_2 \sigma_i, \\ i &= 1, 2, 3, \\ \gamma^5 &= \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \varrho_1. \end{aligned} \quad (2.2)$$

The interaction then has the axial vector form

$$\begin{aligned} H_W &= W \sum_{\mathbf{n}} (\bar{\psi} O^j \psi) (\bar{\psi} O_j \psi), \\ O^j \cdot O_j &= i \gamma^5 \gamma^\mu \cdot i \gamma_5 \gamma_\mu, \quad \bar{\psi} \equiv \psi^\dagger \gamma_0. \end{aligned} \quad (2.3)$$

In the following we shall use the metric $g^{00} = +1$, $g^{ii} = -1$, $i = 1, 2, 3$.

⁴ W. HEISENBERG, Proc. Conf. High Energy Physics, CERN, Geneva 1958.

⁵ E. G. WEIDEMANN, Z. Naturforsch. **24 a**, 499 [1969].

As pointed out above, the concept of a lattice of Z^3 points is a natural way of explaining the cutoff necessary in calculations for nonrenormalizable interactions. Since we shall be using the lattice formulation in the following, it may help the reader to have also the rules for transcribing any formula into the corresponding cutoff-dependent continuum formulation. These are (WEIDEMANN⁵) if we set $l=1$, $b=2\pi/Za$,

$$\begin{aligned}\psi(\mathbf{x}) &\longleftrightarrow a^{-3/2} \psi(\mathbf{n}), \\ \delta(\mathbf{x}-\mathbf{x}') &\longleftrightarrow a^{-3} \delta_{\mathbf{n}\mathbf{n}'}, \\ \int d\mathbf{x} &\longleftrightarrow a^3 \sum_{\mathbf{n}}, \\ g &\longleftrightarrow W a^3, \\ \tilde{\psi}(\mathbf{p}) &\longleftrightarrow b^{-3/2} \tilde{\psi}(\mathbf{h}), \\ \delta(\mathbf{p}-\mathbf{p}') &\longleftrightarrow b^{-3} \delta_{\mathbf{h}\mathbf{h}'}, \\ \int d\mathbf{p} &\longleftrightarrow b^3 \sum_{\mathbf{h}}, \\ S_F(\mathbf{x}-\mathbf{x}') &\longleftrightarrow a^{-3} S_F(\mathbf{n}-\mathbf{n}'),\end{aligned}\quad (2.4)$$

g being the coupling constant in the continuum formulation. Here \mathbf{h} is the vector in the reciprocal lattice, defined for instance by the Fourier transformation

$$\psi(\mathbf{n}) = \frac{1}{\sqrt{Z^3}} \sum_{\mathbf{h}} \tilde{\psi}(\mathbf{h}) \exp \frac{2\pi i \mathbf{h} \cdot \mathbf{n}}{Z}.$$

It is related to the momentum \mathbf{p} by the relation¹

$$p_i = \sqrt{Z} \sin(2\pi/Z) h_i. \quad (2.5)$$

It is now convenient to define the four-vectors

$$n^\mu = (t, \mathbf{n}), \quad p^\mu = (\omega, \mathbf{p}), \quad h^\mu = \left(\omega, \frac{2\pi}{Z} \mathbf{h} \right), \quad (2.6)$$

where $\omega = \sqrt{\mathbf{p}^2 + m^2}$ for a particle of mass m .

It will be useful for comparison purposes to discuss also the interaction considered by NAMBU and JONA-LASINIO⁶⁻⁸ for which

$$0 \cdot 0 = \gamma^5 \cdot \gamma_5 - 1 \cdot 1. \quad (2.7)$$

Both (2.3) and (2.7) are seen to be invariant under the chirality transformation

$$\psi \rightarrow \exp[i\alpha\gamma_5]\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp[i\alpha\gamma_5]. \quad (2.8)$$

H_0 is also invariant. Thus, adding a term $m\bar{\psi}\psi$ to H_0 and subtracting the same term from H_W , the

overall invariance of H remains unaffected. However, considering this *modified interaction* Hamiltonian as a perturbation on the *modified free* Hamiltonian, we may find perturbation solutions which violate the invariance under (2.8). Further, by imposing a condition on the perturbation terms, we may even fix the value of m and interpret it as the selfenergy of the urfermions and so as the mass of a quasiparticle. A natural condition to impose is that the contributions of the selfenergy Feynman diagrams to the one-particle Green's function add up to the physical quasiparticle mass. Considering only diagrams of the first order in the coupling constant — i.e. the Hartree and Fock diagrams shown in Fig. 1 (we are here using the diagrammatization proposed by LURIÉ and MACFARLANE⁹ although the diagrams of quantum electrodynamics would do equally well⁵) — we obtain the Hartree-Fock equation of the quasiparticle selfenergy m . This may immediately be read off the diagrams with the help of Feynman rules:

Fig. 1.
Hartree and Fock diagrams.



$$m = -2iW O_j [\text{Tr}(S_F(-0) O^j) - S_F(0) O^j]. \quad (2.9)$$

Here S_F is the propagator of the free fermion of mass m , and the continuum definitions of the S_F 's are

$$S_F(0) = [S_F(+0) + S_F(-0)], \quad (2.10)$$

$$S_F(\pm 0) = \frac{1}{2i} \int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{m \pm \gamma_0 \sqrt{\mathbf{p}^2 + m^2}}{\sqrt{\mathbf{p}^2 + m^2}} \right).$$

Substituting in (2.9) and using the lattice space formulation we obtain for *either* of the interactions (2.3), (2.7) the condition

$$m = m \frac{4W}{Z^3} \sum_{\mathbf{h}} \frac{1}{\sqrt{\mathbf{p}^2 + m^2}}, \quad \mathbf{p} = \mathbf{p}(\mathbf{h}). \quad (2.11)$$

Thus, besides the trivial solution¹⁰ $m=0$ this self-consistency condition contains another solution $m \neq 0$ which violates the invariance of H_0 and H_W under (2.8). The Goldstone theorem then predicts the concurrent existence of a massless boson.

⁶ Y. NAMBU, Proc. Rochester Conf. 1960, p. 858.

⁷ Y. NAMBU and G. JONA-LASINIO, Phys. Rev. **122**, 345 [1960].

⁸ Y. NAMBU and G. JONA-LASINIO, Phys. Rev. **124**, 246 [1961].

⁹ D. LURIÉ and A. J. MACFARLANE, Phys. Rev. **136 B**, 816 [1964].

¹⁰ Strictly speaking $m=0$ is a solution of (2.11) only in the continuum formulation. In the lattice space formulation the solution is $m=O(W/Z)$. I am indebted to Dr. FRIEDEL for emphasizing this point.

3. Solution of the Homogeneous Bethe-Salpeter Equation

We next investigate the BS-equation for bound states of a quasiparticle-antiquasiparticle pair. The BS-amplitude for the binding of this pair of four-momenta p_1 , $-p_2$ to a bound state of four-momentum q is the expectation value

$$\Phi = \langle 0 | T \psi(p_1) \bar{\psi}(-p_2) | q \rangle. \quad (3.1)$$

The BS-equation of which Φ is the solution may now be derived in the customary way¹¹. Moreover, once the form of this equation is known, one can invent diagrams which yield this equation when translated by means of Feynman rules. This is achieved by using instead of Φ the vertex-function Γ defined by the relation

$$\Phi = S_F(p_1) \Gamma(p_1, p_2) S_F(p_2) \quad (3.2)$$

and represented by a circle in Feynman diagrams. Next the desire for explicit solutions compels us to make drastic approximations. We shall use first only the so-called chain approximation and extend it later to the sum of all mixed and unmixed chain and ladder graphs. The chain approximation con-

sists in summing the diagrams shown in Fig. 2 (the double-lined arrow representing the outgoing bound state).

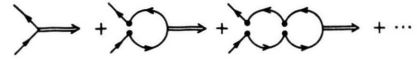


Fig. 2. The chain approximation.

The diagrammatic form of the corresponding homogeneous BS-equation is shown in Fig. 3.

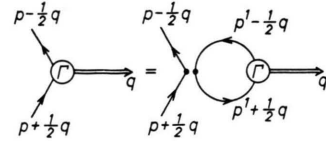


Fig. 3. The homogeneous BS-equation in the chain approximation.

Since we are primarily interested in the mass of the outgoing bound state, it is convenient to use a frame of reference in which this particle is at rest, i. e. $q = (E, 0)$, where E is the total energy of the system. The four-momenta of the quasiparticle-antiquasiparticle pair may then be reexpressed in terms of their relative momentum p :

$$p_1 = p + \frac{1}{2} q, \quad -p_2 = \frac{1}{2} q - p, \quad \mathbf{p} = \sqrt{Z} \sin(2\pi \mathbf{h}/Z). \quad (3.3)$$

The Feynman rules now yield immediately the equation

$$\Gamma(p + \frac{1}{2} q, p - \frac{1}{2} q) = -2 \left(-\frac{iW}{Z^3} \right) C \sum_{\mathbf{k}} \int \frac{dp'_0}{2\pi} \cdot \text{Tr}[O i S_F(p' + \frac{1}{2} q) \Gamma(p' + \frac{1}{2} q, p' - \frac{1}{2} q) i S_F(p' - \frac{1}{2} q)] \quad (3.4)$$

Here the factor 2 comes from a doubling of time-ordered products for the fourfermion interaction so that every W is accompanied by a factor 2. The first minus sign results from the closed fermion loop, and $-iW$ from the perturbation expansion. Moreover every fermion line carries a factor $i S_F/(2\pi)^4$ and the delta function of energy-momentum conservation carries a further $(2\pi)^4$. We observe from (3.1), (3.2) that the vertexfunction Γ transforms in the spin space of the two-particle system as the two-particle amplitude. Thus if the latter transforms as a pseudoscalar, i. e. if the composite state is a pseudoscalar particle, then Γ must also transform that way. Setting therefore in (3.4) $\Gamma = a \gamma_5$, a a momentum-independent constant, we obtain an equation determining q , the mass of that particle. We consider first the interaction (2.7) of the *Nambu model*. Equation (3.4) is readily seen to simplify to

$$1 = -\frac{8iW}{Z^3} \sum_{\mathbf{k}} \int_C \frac{dp_0}{2\pi} \cdot \frac{(m^2 - p_0^2 + \mathbf{p}^2 + \frac{1}{4} q^2)}{(p_0 - p_{01})(p_0 - p_{02})(p_0 - p_{03})(p_0 - p_{04})}, \quad (3.5)$$

where $p_{01,2} = +(\sqrt{\mathbf{p}^2 + m^2} \pm \frac{1}{2} q) = -p_{03,4}$, and C is the contour shown in Fig. 4.

The poles have the relative positions shown there since for pure binding we must have $\frac{1}{2} q < \sqrt{m^2 + \mathbf{p}^2}$ for all \mathbf{p} . Integration now yields

$$1 = \frac{4W}{Z^3} \sum_{\mathbf{k}} \frac{1}{\sqrt{\mathbf{p}^2 + m^2}} [1 - \frac{1}{4} q^2 / (\mathbf{p}^2 + m^2)]^{-1}. \quad (3.6)$$

¹¹ D. LURIE, *Particles and Fields*, John Wiley, New York 1968.

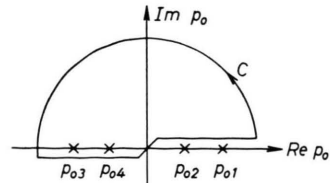


Fig. 4. Integration contour for (3.5).

For $q=0$ this equation reduces to the selfconsistency condition (2.11), thus verifying the existence of the massless Goldstone boson — here a pseudo-scalar bound state.

If we search for a scalar particle, we set

$$\Gamma = \text{const} \times 1$$

in (3.4). Repeating the above arguments one now finds

$$1 = \frac{4W}{Z^3} \sum_{\mathbf{k}} \mathbf{p}^2 / \sqrt{\mathbf{p}^2 + m^2} (\mathbf{p}^2 + m^2 - \frac{1}{4} q^2). \quad (3.7)$$

For $q^2 = 4m^2 - \varepsilon_0$, $\varepsilon_0 > 0$ this equation reduces to (2.11). Thus the theory contains a scalar bound state of mass $2m - \varepsilon$. However, (3.4) does not possess solutions consistent with (2.11) when Γ transforms as a vector or axialvector. For this reason

The Feynman rules then yield the equation

$$\Gamma(p + \frac{1}{2}q, p - \frac{1}{2}q) = \frac{2(-iW)}{Z^3} \sum_{\mathbf{k}'} \int \frac{dp'_0}{2\pi} [O i S_F(p' + \frac{1}{2}q) \Gamma(p' + \frac{1}{2}q, p' - \frac{1}{2}q) i S_F(p' - \frac{1}{2}q) 0]. \quad (3.8)$$

To simplify this equation we use the Fierz formula

$$F_{\gamma\alpha} G_{\sigma\mu} = \frac{1}{4} \sum_{A=1}^{16} \gamma_{\sigma\alpha}^{[A]} (F \gamma^{[A]} G)_{\gamma\mu}. \quad (3.9)$$

Here F, G are arbitrary matrices and the $\gamma^{[A]}$'s are 16 matrices having the properties

$$(\gamma^{[A]})^2 = 1, \quad \text{Tr}(\gamma^{[A]}) = 0 \quad \text{except for } A = 1, \quad \text{Tr}(\gamma^{[A]} \gamma^{[B]}) = 0 \quad \text{for } A \neq B.$$

We shall use the following representation

$$\begin{aligned} \gamma^{[1]} &= 1; \gamma^{[2,3,4,5]} = \gamma^0, i\gamma^j; \gamma^{[6-11]} = \gamma^0 \gamma^j, i\gamma^1 \gamma^2, i\gamma^2 \gamma^3, i\gamma^3 \gamma^1; \\ \gamma^{[12-15]} &= i\gamma^5 \gamma^0, \gamma^5 \gamma^j; \gamma^{[16]} = \gamma^5; \quad j = 1, 2, 3. \end{aligned} \quad (3.10)$$

Considering the matrices 0·0 in (3.8) in the form (3.9) and inserting the interaction (2.7), we obtain a BS-equation of the chain type:

$$\begin{aligned} \Gamma(p + \frac{1}{2}q, p - \frac{1}{2}q) &= \frac{iW}{Z^3} \sum_{\mathbf{k}'} \int \frac{dp'_0}{2\pi} \gamma^\mu \text{Tr}[\gamma_\mu i S_F(p' + \frac{1}{2}q) \Gamma(p' + \frac{1}{2}q, p' - \frac{1}{2}q) i S_F(p' - \frac{1}{2}q)] \\ &\quad - \frac{iW}{Z^3} \sum_{\mathbf{k}'} \int \frac{dp'_0}{2\pi} \gamma^5 \gamma^\mu \text{Tr}[\gamma_5 \gamma_\mu i S_F(p' + \frac{1}{2}q) \Gamma(p' + \frac{1}{2}q, p' - \frac{1}{2}q) i S_F(p' - \frac{1}{2}q)]. \end{aligned} \quad (3.11)$$

If we now set $\Gamma = \text{const} \times \gamma^2$ and proceed as above, we obtain after integration over p'_0 :

$$1 = f_N(q^2) \equiv \frac{4W}{Z^3} \sum_{\mathbf{k}} \frac{\frac{1}{2} (3m^2 + 2\mathbf{p}^2)}{\sqrt{\mathbf{p}^2 + m^2} [4(m^2 + \mathbf{p}^2) - q^2]}. \quad (3.12)$$

Plotting $f_N(q^2)$ versus q^2 for $W > 0$, we obtain a curve whose upper limit is shown in Fig. 7.

The theory therefore contains a vector bound state of mass q with $\sqrt{2}m \leq q \leq 2m$. For finite Z this value of q is less than $2m$. In the same manner

we now consider the sum of ladder graphs shown in Fig. 5.

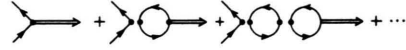


Fig. 5. The ladder approximation.

The diagrammatic form of the corresponding vertex equation is shown in Fig. 6.

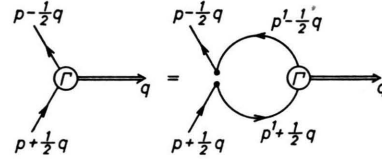


Fig. 6. The homogeneous BS-equation in the ladder approximation.

one finds that the theory does not contain an axial-vector bound state.

We next discuss the *Bopp model* interaction (2.3). It is readily seen that the pure chain approximation leads to an axialvector bound state.

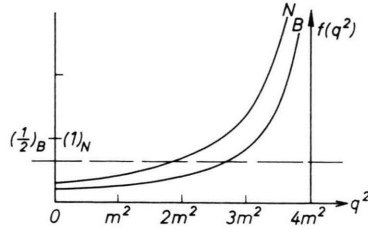


Fig. 7. Upper limits of the curves $f_{N,B}(q^2)$ for $W > 0$.

The vertex equation in the pure ladder approximation may again be converted into one of the type of a chain approximation by using the Fierz formula. We find

$$\begin{aligned} \Gamma(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) \\ = -\frac{2iW}{Z^3} \sum_{k'} \int \frac{dp'_0}{2\pi} P \operatorname{Tr}[P_i S_F(p' + \tfrac{1}{2}q) \\ \cdot \Gamma(p' + \tfrac{1}{2}q, p' - \tfrac{1}{2}q) i S_F(p' - \tfrac{1}{2}q)] \end{aligned}$$

where

$$P \cdot P = 1 \cdot 1 + \tfrac{1}{2} \gamma^\mu \cdot \gamma_\mu + \tfrac{1}{2} \gamma^5 \gamma^\mu \cdot \gamma_5 \gamma_\mu - \gamma^5 \cdot \gamma_5. \quad (3.13)$$

Setting $P = \gamma_5 \times \text{const}$ we again obtain (3.6). Thus the Bopp model contains a massless pseudoscalar bound state, the Goldstone boson. In a similar manner (3.13) reduces to (3.7) for a scalar particle of mass $2m - \varepsilon$, $\varepsilon > 0$. For a vertex function transforming as a vector γ^2 (3.13) reduces to

$$1 = -\frac{4W}{Z^3} \sum_k \frac{p^2 + \frac{3}{2}m^2}{\sqrt{p^2 + m^2} [4(p^2 + m^2) - q^2]}. \quad (3.14)$$

There is no value of q in the range $0 \leq q < 2m$ for which this equation reduces to (2.11) (which would be necessary for consistency). Hence the Bopp model does not contain a vector bound state. However, it contains an axialvector bound state. In this case we find that (3.13) reduces to

$$1 = f_B(q^2) \equiv \frac{4W}{Z^3} \sum_k \frac{m^2/2 - p^2}{\sqrt{p^2 + m^2} [4(m^2 + p^2) - q^2]}. \quad (3.15)$$

Plotting $f_B(q^2)$ versus q^2 for $W > 0$ we obtain a curve whose upper limit is shown in Fig. 7. Thus the Bopp model contains an axialvector bound state of mass q in the range $\sqrt{3}\frac{1}{2}m < q \leq 2m$; for finite Z this mass lies well below $2m$.

We now wish to extend these calculations to the sum of all mixed and unmixed chain and ladder diagrams. One can readily convince oneself of the following theorem [e. g. by an inductive procedure, considering all Feynman diagrams of first, second, ... order and using the Fierz formula¹² (3.9)]: Let

the given fourfermion matrix interaction of the type (2.3) be $V \cdot V$. Then the contribution of the sum of all mixed and unmixed vertex diagrams of any order (having interaction $V \cdot V$ at each vertex) is equal to the contribution of the pure chain vertex diagram for the interaction

$$0 \cdot 0 = V \cdot V - W \cdot W \quad (3.16)$$

at each vertex, where $W \cdot W$ is the Fierz transform of $V \cdot V$. As an example we consider the sum A of first-order vertex diagrams shown in Fig. 8.

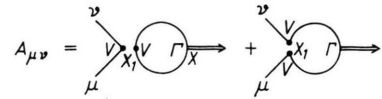


Fig. 8. First-order vertex diagrams.

The Fierz formula (3.9) allows the substitution

$$V_{\alpha\beta} V_{\gamma\delta} = W_{\gamma\beta} W_{\alpha\delta}, \quad (3.17)$$

where $W \cdot W$ is the Fierz transform of $V \cdot V$. Hence

$$\begin{aligned} A_1 &= -V_{\mu\nu} \operatorname{Tr}(V S_F(x_1 - x) \Gamma S_F(x - x_1)) \\ &\quad + V_{\mu\varrho} (S_F(x_1 - x) \Gamma S_F(x - x_1))_{\varrho\kappa} V_{\kappa\nu} \\ &= -V_{\mu\nu} \operatorname{Tr}(V S_F(x_1 - x) \Gamma S_F(x - x_1)) \\ &\quad + W_{\mu\nu} \operatorname{Tr}(W S_F(x_1 - x) \Gamma S_F(x - x_1)) \end{aligned}$$

as claimed by (3.16).

We now ask: How are our previous results on bound states affected if we sum over all mixed and unmixed chain and ladder diagrams? The diagrammatic form of the BS-equation is now given by Fig. 3 except that $0 \cdot 0$ is to be replaced by (3.16). In the case of the Nambu model — where by (2.7) we have $V \cdot V = \gamma_5 \cdot \gamma_5 - 1 \cdot 1$ — we now have

$$0 \cdot 0 = \gamma_5 \cdot \gamma_5 - 1 \cdot 1 + \tfrac{1}{2} [\gamma^\mu \cdot \gamma_\mu - \gamma^5 \gamma^\mu \cdot \gamma_5 \gamma_\mu] \quad (3.18)$$

and in the Bopp model

$$0 \cdot 0 = \gamma_5 \cdot \gamma_5 - 1 \cdot 1 - \tfrac{1}{2} [\gamma^\mu \cdot \gamma_\mu + 3 \gamma^5 \gamma^\mu \cdot \gamma_5 \gamma_\mu]. \quad (3.19)$$

It is readily seen that the previous results in the Nambu model remain unchanged. However in the Bopp model, both chain and ladder diagrams contribute to the axialvector state, i. e. to the factor $3/2$ in (3.19). Thus the previous results concerning scalar, pseudoscalar and vector states remain unchanged. In the case of the axialvector bound state the coupling constant W in (3.15) is replaced by $3W$ or, alternatively, the 1 on the left of that equation by $1/3$. The solution therefore lies somewhere along the line $f_B = 1/3$ in Fig. 7 depending on the value of Z .

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4. Some Remarks on the Scattering Problem

Finally we add some remarks on the scattering problem in order to obtain information on the Z -behaviour of coupling constants and cross sections. For scattering of a quasiparticle-antiquasiparticle pair it is necessary to solve the inhomogeneous BS-equation. The diagrammatic form of this equation in the chain approximation is shown in Fig. 9, where the box represents the scattering amplitude T . It is again convenient to use relative and centre of mass

momenta p, q and corresponding reciprocal lattice vectors h, g . The equation for T is then seen to be

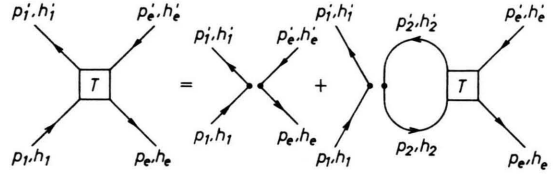


Fig. 9. The inhomogeneous BS-equation in chain approximation.

$$O_j T(h, g) \cdot O^j = -2iW a^3 O_j \cdot O^j + 2iW O_j \int \frac{dh'_0}{2\pi} \frac{1}{Z^3} \sum_{\mathbf{h}'} \text{Tr}[O^j i S_F(h' + \frac{1}{2}g) O_j T(h', g) i S_F(h' - \frac{1}{2}g)] \cdot O^j, \quad (4.1)$$

where we have set

$$p_1, p'_1 = p \pm \frac{1}{2}q, \quad p_2, p'_2 = p' \pm \frac{1}{2}q, \quad h_1, h'_1 = h \pm \frac{1}{2}g, \quad h_2, h'_2 = h' \pm \frac{1}{2}g. \quad (4.2)$$

Assuming now that T depends only on g , and inserting in (4.1) the interaction (2.7), we obtain

$$T(g) \left[\frac{1}{f} \gamma_5 \cdot \gamma_5 - \frac{1}{e} 1 \cdot 1 \right] = - \frac{2iW a^3}{ef} [\gamma_5 \cdot \gamma_5 - 1 \cdot 1], \quad (4.3)$$

where

$$e \equiv 1 - 2iW \int \frac{dh'_0}{2\pi} \frac{1}{Z^3} \sum_{\mathbf{h}'} \text{Tr}[\gamma_5 i S_F(h' + \frac{1}{2}g) \gamma_5 i S_F(h' - \frac{1}{2}g)], \quad (4.4)$$

$$f \equiv 1 + 2iW \int \frac{dh'_0}{2\pi} \frac{1}{Z^3} \sum_{\mathbf{h}'} \text{Tr}[i S_F(h' + \frac{1}{2}g) i S_F(h' - \frac{1}{2}g)].$$

The scattering amplitude therefore contains parts transforming as pseudoscalars and scalars in spin space. In particular the pseudoscalar part is seen to have a pole at $e=0$ and the scalar part at $f=0$. The equations $e=0, f=0$ are readily seen to be identical with our previous vertex equations in chain approximation for T proportional to γ_5 and 1 respectively, thus reaffirming that the bound states represent poles of the scattering amplitude. We are now interested in the "coupling constant" G^2 between the quasiparticles and (e. g.) the pseudoscalar state. This is, of course, given by the residue of T at the pseudoscalar bound state pole — or (which is the same in our case) by the semiphenomenological equivalence of a chain of pseudoscalar chain diagrams with a single line representing the pseudoscalar bound state. For simplicity we set $g=0$. Then if we expand e in the neighbourhood of the pole $q^2=0$ and use the selfconsistency condition (2.11) we obtain

$$e = q^2 \frac{W}{Z^3} \sum_{\mathbf{h}} \frac{1}{(\mathbf{p}^2 + m^2)^{3/2}} \sim W a^3 \ln Z. \quad (4.5)$$

The equivalence

$$T \gamma_5 \cdot \gamma_5 = - \frac{2iW a^3}{e} \gamma_5 \cdot \gamma_5 = -iG \frac{1}{q^2} G \quad (4.6)$$

now yields

$$G^2 = \frac{2a^3}{\frac{1}{Z^3} \sum_{\mathbf{h}} \frac{1}{(\mathbf{p}^2 + m^2)^{3/2}}} \quad (4.7)$$

$$= 2 \left[\frac{1}{(2\pi)^3} \int^A \frac{d\mathbf{p}}{(\mathbf{p}^2 + m^2)^{3/2}} \right]^{-1}.$$

Thus G^2 has the behaviour of $(\ln Z)^{-1}$. It is important to observe that this behaviour is independent of the original fourfermion coupling constant W and depends only on the cutoff Λ or equivalently Z . However W , and hence the Z -dependence of W , are important in determining the Z -behaviour of an amplitude and the differential cross-section. The Z -behaviour of an amplitude is easily found in the momentum representation. E. g. the scattering process corresponding to a Feynman diagram of the fourth order contains four vertices contributing together a factor W^4 , six internal lines contributing

in all $(1/\sqrt{Z})^6$ and summations and integrations over three independent momenta contributing

$$\frac{1}{Z^3} \sum_{\mathbf{h}} \int d\mathbf{h}_0 \sim (\sqrt{Z})^3.$$

The amplitude therefore behaves as

$$W^4(\sqrt{Z})^3/(\sqrt{Z})^6 \sim \sqrt{Z} \text{ if } W \sim \sqrt{Z}$$

as in Bopp's theory. By similar considerations one finds that the amplitude corresponding to a Feynman diagram of any order behaves as \sqrt{Z} . The corresponding differential scattering cross-section $d\sigma/d\Omega$ is found to behave as $1/Z^2$. This behaviour becomes somewhat milder if the amplitude is a

chain of diagrams, because then [using (4.5) as an example] the scattering cross-section has to be divided by e^2 and so behaves as $1/e^2 Z^2 \sim (\ln Z)^{-2}$. We see therefore that the Z -dependence of W which was chosen³ to ensure a finite, noninfinite mass of the quasiparticles results in vanishing cross-sections at least in the limit $Z \rightarrow \infty$. This may best be understood by observing¹⁴ that in a continuum theory this procedure corresponds to the replacement $g \rightarrow g/\Lambda^2$, where Λ is the cutoff.

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Impurity-induced and Second Order Raman Spectra of NaCl Crystals Doped with Different Ag^+ Concentrations

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Raman spectra of NaCl crystals doped with Ag^+ ions up to 3.5 mol% have been measured in 4 scattering geometries. In the impurity induced part of the E_g - and F_{2g} -spectra the strongest peaks are located at 85 and 171 cm^{-1} . Their intensities increase proportional to the silver concentration. The second order Raman spectrum on the other hand appears to be independent of the Ag^+ content. The spectra are fully explained by theory and by the assumption of Ag^+ pairs.

Introduction

Besides well known methods like optical absorption and thermal conductivity spontaneous Raman scattering recently has proven as a useful tool for investigating the influence of point imperfections on the vibrations of the host lattice. If such imperfections are introduced into a cubic alkali halide crystal the translational symmetry of the lattice is disturbed. Then not only the two-phonon Raman spectrum which is alone allowed in the pure crystal

but also one phonon scattering may be expected. This scattering is in general characteristic of both the host lattice and the particular imperfections.

In recent years the influence of various kinds of lattice imperfections such as atomic impurities¹⁻⁴, diatomic and molecular impurities⁵⁻⁹ as well as F-centers^{10, 11} has been investigated in alkali halide crystals. First order Raman effect could be clearly established in KBr crystals, doped with Cl^- ions¹, in NaCl doped with Ag^+ ³ and in three potassium halides doped with Ti^{4+} ⁴.

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